Math Overboard

Three Checkers Left and Variations

By Art Benjamin

I still remember the first time my name appeared in a backgammon book. It was nearly thirty years ago, and I had sent Danny Kleinman some formulas for cube actions in the last stages of the bearoff, which he dubbed “Benjamin’s Benchmarks,” and published them in his book, *But Only the Hogs Win Backgammons*, in 1991.

The benchmarks covered endgame cube actions where one player had a no-miss position (all checkers on the ace point). In this article, we will explore variations on this theme. In the spirit of Chess, we will refer to the player on roll as White and the opponent as Black.

Over the years, I have probably applied the simplest benchmark, which I call the **11-12 Rule**, thousands of times. For a 3-checker position, let’s define the Simply Adjusted Pip Count (SAPC) as the Pip Count with a 1-pip penalty if there are any stacked checkers. For example, position 542 (one checker apiece on the 5, 4, and 2 points) has SAPC = 11, but position 443 has SAPC = 12.

3 checkers versus 2-roll no-miss position

The **11-12 rule**: If White has 3 checkers left, and Black has 3 or 4 checkers on their ace point. The Double/Take Window is when White has SAPC = 11 or 12.

In other words, If SAPC ≤ 10, then the cube action is Double/Pass. If SAPC = 11 or 12, it is Double/Take. If SAPC ≥ 13, then it is No Double/Take. Every position that would be an initial double is also a redouble.

For practice, determine the cube action for positions 442, 542, and 552, as shown in Figure 1.

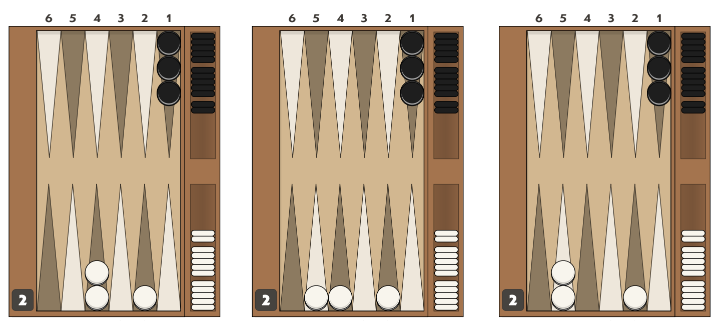

Fig. 1. What are the cube actions for these positions?

Positions 442 and 542 have SAPC = 11, so they would be Redouble/Take. Position 552 has SAPC = 13, so it would not be a double. The 11-12 Rule is extremely accurate, and there are no serious exceptions to it. Position 333 has SAPC = 10, and although it is a borderline take, the error from dropping is only 0.01.

We note that when SAPC = 10, Black's winning chances are about 22%, and since Black has no cube leverage (unless White rolls 21), it's a 0.10 error to take the cube. When White has SAPC = 9, 10, 11, 12, and 13, then Black's cubeless chance of winning is about 18%, 22%, 29%, 35% and 42%, respectively. As a result, the 11-12 Rule will hold up at most match scores that are not extremely lopsided.

4 checkers versus 2-roll no-miss position

What if we give White one more checker so the player on roll has 4 checkers instead of 3? Black still has position 111. Unless there is at least one checker on the ace point, it is no double. Indeed, if doubled, Black would have a proper Beaver when White has position 2222, as shown in Figure 2.

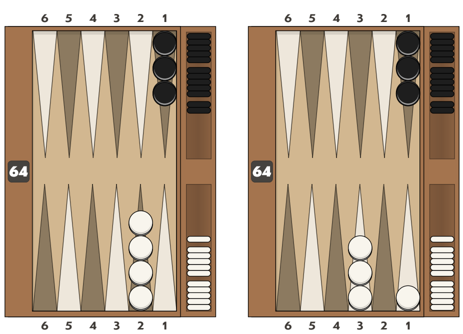

Fig. 2. If White doubles from 2222 or 3331 then Black should Beaver.

The best reference for this situation is that 4321 is a clear Redouble/Take with Black winning 26% of the time. Move the checker on the 4 point back to the 5 point and position 5321 is Double/Take but not a Redouble. Move that checker forwards and 3321 becomes only a borderline take (with Black winning 24.6% of the time). Moving that checker forwards to 3221 makes it a clear pass with Black winning just 21.5% of the time. White can still redouble the gappy positions 4221 and 3311, but dares not double with 3331 since all twos are lethal and all aces make White an underdog who will face an immediate recube from Black.

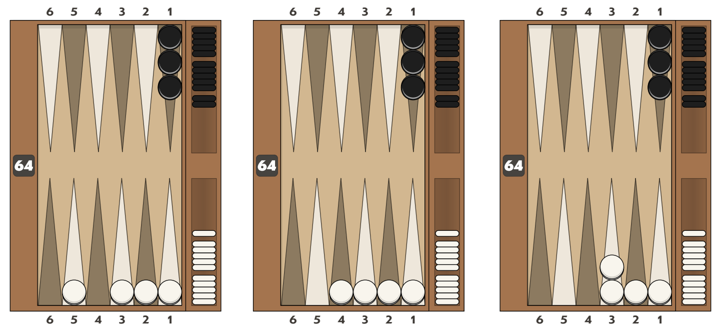

Fig. 3. Position 5321 is a double, but not a redouble; 4321 is a solid redouble and take; 3321 is a redouble and borderline take.

Be forewarned that the 4321 position is a pass for any other collection of 3 checkers (except the equivalent 211) since it requires that Black have all 6 working doubles. For example, 4321 versus 321 is a close pass with Black winning 24%.

3 or 4 checkers versus no-miss 3-roll position

Now let us give Black a no-miss 3 roll position, so Black has 5 or 6 checkers on the ace point. If White has 3 checkers, then White should always Redouble and Black should Pass unless White has unlikely position 665 or 666.

When White has 4 checkers, then the situation is more interesting, and the doubling action can be determined by White's Pip Count (with no adjustments needed). I call it the **Liesl Rule** from the character in *The Sound of Music* who sings the song "Sixteen Going on Seventeen."

The **Liesl Rule**: If White has 4 checkers left, and Black has 5 or 6 checkers on their ace point. *The Double/Take window is when White has a Pip Count of 16 or 17.* All doubles are also redoubles.

Thus, when White has a Pip Count of 15 or less, then Black must pass. With a Pip Count of 16 or 17, it is Redouble/Take, and when White has 18 or more pips, then it is not a double or redouble.

Try your hand at the following positions.

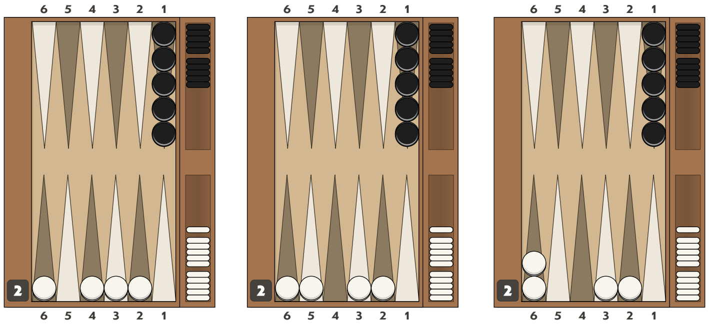

Fig. 4. Use the Liesl Rule for the following positions

Position 6432 has a Pip Count of 15, so it would be Double/Pass; Position 6532 has a Pip Count of 16, so it would be Double/Take, and position 6632 has a Pip Count of 17, so it would also be Double/Take. Note that the Liesl Rule, as stated, makes no penalty for stacked checkers.

In most racing situations, prior to the last few rolls, at the point of last take, the trailer's game winning chances are somewhere between 22% and 24% percent. But here, this is the range for the point of first pass. For instance, when White has Pip Count = 15, Black's winning chances are 23%. Within 2 pips of that, each pip changes the winning chance by about 4%. In other words, with pip counts 13, 14, 15, 16, 17, and 18, Black's winning chances are 15%, 19%, 23%, 27%, 31% and 35%, respectively.

There are some exceptions to this rule, the main one being that the position 6543, despite having 18 pips, is still a redouble. For better accuracy, we introduce the Adjusted Pip Count in the next section. But before diving into that, I recommend pausing to make sure that you have everything in this section solid. If you try to absorb too much in one sitting, you might wind up retaining very little.

The Adjusted Pip Count (APC)

We define the **Adjusted Pip Count (APC)** as the pip count plus a half-pip penalty for every stacked checker on the 6, 5, 4, or 3 point, a 1-pip penalty for every stacked checker on the 2 point, and a 2-pip penalty for every stacked checker on the 1 point. (For those familiar with other racing formulas, the penalties applied to the ace and deuce points are the same as those applied by the Keith Count.) Although we are only applying the APC in this article to positions with a small number of checkers, here are the penalties that will be most relevant for this article.

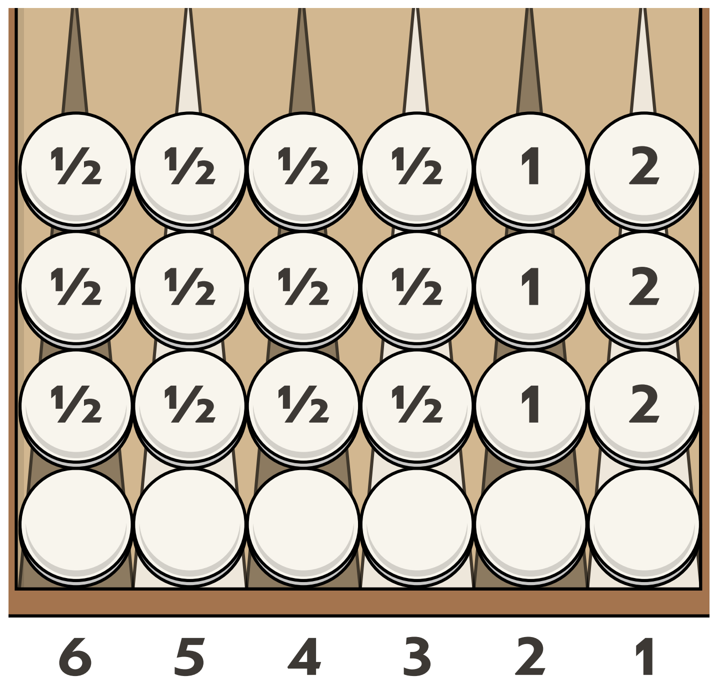

Fig. 5. Using the APC, stacked checkers are penalized as in the diagram above. Note that we will be applying APC only to positions with 4 checkers or less.

For example, the positions 442, 542, and 552 (shown earlier in Figure 1) would have APC equal to 10.5, 11, and 12.5, respectively. Using APC (instead of SAPC), the 11-12 rule would instead give a Double/Take window of 10.5 to 12, with the same results as before. We note that when White has APC = 12.5, White has an optional double, but not a redouble.

Using APC, the Liesl Rule can be given better accuracy by allowing the redoubling window to encapsulate all positions with

$$16 \le \text{APC} \le 18.$$

For example, if White has position 4443 which has a Pip Count of 15, the original Liesl Rule would suggest that Black (with position 11111) would have to drop White's double. But that would be a mistake. The revised Liesl rule sees that 4443 has an APC of 16 (since the second and third checker on the 4 point each contribute 0.5) and therefore correctly suggests that Black can take White's double. Likewise, if White has position 6522, this also has APC = 16, which allows Black to take White's double as well.

3-roll no miss position versus 3 or 4 checkers

What about the reverse situation? Now suppose that White, on roll, has 5 (or 6) checkers on the ace point, and Black has 3 or 4 checkers. What are the correct cube actions here?

If Black has just 3 checkers, then White should double only if Black has APC ≥ 15. (As we'll soon see, this is precisely when Black is favored to need three or more rolls to bear off.) Black can take except in the seldom seen devilish position 666.

When Black has 4 checkers, Black is much more likely to need three rolls, and now White's Double/Take window is

$$12.5 \le \text{APC} \le 15$$

I'll call this the **Louisa Rule**, for Liesl's younger sibling! That is, if Black has APC ≤ 12, then White should not double. If APC > 15, then Black should Pass. With rare exceptions (when White has APC = 12.5 or 13), all doubles are redoubles.

All 56 possible 3-checker positions are presented in order of increasing APC in Table 1 below. For the lowest APC values, say from 6 to 8, there is hardly any difference between the positions. (For example, position 321 is in fact slightly worse than 111, despite having a lower APC value). We also include the Effective Pip Count (EPC) which is the average number of pips needed to bear off that position. We will have much more to say about EPC in future articles, but for 3-checker positions with 10 or more pips, a good approximation is to add 6 to the pip count.

Table 1. The 3-roll positions, with their pip counts, APC, EPC, and the percent chance of failing to bear off in two rolls.

Position	PC	APC	EPC	P(fail2)
211	4	6	15.0	0
221	5	6	15.2	0
321	6	6	15.2	0
111	3	7	15.0	0
311	5	7	15.2	0
421	7	7	15.2	0.3
331	7	7.5	15.5	0.3
222	6	8	15.2	0
411	6	8	15.2	0
322	7	8	15.2	0.3
431	8	8	15.6	1.5
521	8	8	15.6	2.5
332	8	8.5	15.7	3.1
422	8	9	15.4	2.8
511	7	9	15.5	0.3
531	9	9	15.8	4.1
432	9	9	15.8	4.9
621	9	9	15.9	5.3
441	9	9.5	16.4	9.2
611	8	10	15.9	6.2
631	10	10	16.1	8.7
532	10	10	16.2	9
522	9	10	16.3	10.3
541	10	10	16.3	8.4
333	9	10	16.4	11.8
433	10	10.5	16.5	13
442	10	10.5	16.6	11.2
632	11	11	17.0	19.1
542	11	11	17.0	16.4
622	10	11	17.0	19.2
641	11	11	17.1	18.1
533	11	11.5	17.2	21.8
443	11	11.5	17.3	20.4
551	11	11.5	18.0	25.2
543	12	12	17.7	24.1
642	12	12	17.7	24.2
651	12	12	18.0	25.8
633	12	12.5	17.9	30.1
552	12	12.5	18.5	31.3
643	13	13	18.4	32.8
444	12	13	18.5	34.6
652	13	13	19.0	37.6
544	13	13.5	18.7	36.2
553	13	13.5	19.1	38.2
661	13	13.5	20.1	47.6
653	14	14	19.4	41.3
644	14	14.5	19.6	45.9
554	14	14.5	19.7	45
662	14	14.5	20.6	53.3
654	15	15	20.3	50.4
663	15	15.5	21.1	56.9
555	15	16	21.2	60
655	16	16.5	21.5	61.2
664	16	16.5	21.8	62.7
665	17	17.5	22.5	67
666	18	19	23.8	76.6

The last column indicates how often the position fails to remove all three checkers in two rolls. For example, position 632, with APC = 11, fails about 19% of the time. Notice that positions with APC below 9 rarely fail, positions with APC = 10 fail about 10% of the time, and the position with APC = 15 fails about 50% of the time. Interpolating between these last two data points gives us a surprisingly accurate approximation. Specifically, for positions with APC ≥ 9,

$$P(\text{2-Roll Fail}) \approx 8 \times (\text{APC}) - 70$$

For example, positions with APC = 11 fail about 18% of the time, and positions with APC = 12 fail about 26% of the time. This formula is almost always within 2% of the true probability. This sheds light on why the 11-12 Rule (or the APC 10.5-12 version) works so well. When White has an APC of 10.5 or higher, the table shows that White fails at least 11% of the time. When this is added to the roughly 14% chance that Black rolls doubles (after White fails to roll doubles on the first roll) we see that Black has a comfortable take. When White has an APC of 13 or higher, then White fails at least 33% of the time, so combined with Black's chance of rolling doubles, White does not have a double. Indeed, Black could have a beaver!

Any 3 checkers versus any 3 checkers

So now let us apply APC to determine the proper cube action when both players have 3 checkers remaining, anywhere in the inner board. Let White (on roll) have APC = X and Black have APC = Y. The following extension of the 11-12 Rule is highly accurate.

If X < 11, then White redoubles and Black passes. White essentially has a no-miss 2-roll position, so Black must drop.

The only serious exception (which we already know by the 10.5-12 Rule) is that Black can take if X = 10.5 and Black has position 111 or 211.

If X = 11 or 11.5, then White Redoubles and Black can take if their APC is better. That is, Black can take if Y < X.

If X = 12 or 12.5, then White Redoubles and Black can take if their position is at least as good. That is, Black can take if Y ≤ X. (Exception: If X = 12.5, then White can double, but not redouble when Y ≤ 8. Also, When X = 12.5, Black can take if Y = 13.)

If 13 ≤ X ≤ 14.5, then the redoubling window is when

–3 ≤ Y – X ≤ 1. Note that since White is on roll, White is a slight favorite, even when trailing by 3 (adjusted) pips. As the game is so close to conclusion, the rule above says that White can redouble when a slight favorite. Black can take up to 1 pip worse. (Exception: 643 can double, but not redouble, when Y = 9.)

If X ≥ 15, the redoubling window is –2 ≤ Y – X ≤ 1. As indicated by the 8X – 70 formula, when X ≥ 15, White is an underdog to bear off in two rolls, so White needs to be a slightly better favorite to double or redouble. Black can still take up to 1 pip worse.

Let's practice the extended 11-12 Rule with some examples.

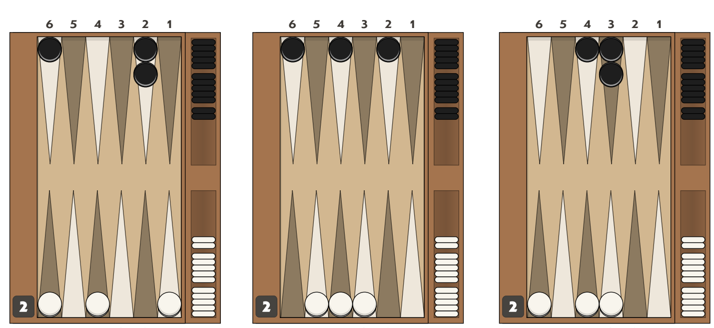

Fig. 6. Determine the cube actions for the above positions

In the first example, X = 11 and Y = 11, since Black is not strictly better than White, this would be a Pass. In the second example, both players have APC = 12, so Black can take White's Redouble. In the last position X = 13 and Y = 10.5. Since Black is within 3 adjusted pips of Black, White can Redouble and Black has an easy Take.

Again, these rules have some exceptions, but most of the errors are not too serious. A calculation of Black's exact winning chances of winning indicates that the threshold for taking is about 23.7%.

3 checkers versus 4 checkers

When Black has a fourth checker, that worsens their situation considerably, not just from the added pips, but from the increased chance of failing to bear off in two rolls. Consequently, we adjust Black's APC by applying a penalty of 1.5 pips, then following the rules in the 3 vs 3 situation. See the examples in Figure 7.

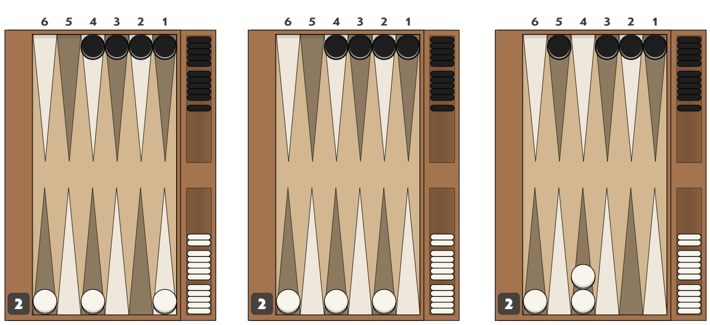

Fig. 7. Determine the cube actions for the above positions

In the first position X = 11 and Y = 10, but adding 1.5 penalty for the fourth checker, brings it to 11.5, so Black has to pass White's redouble. In the second position, X = 12, so Black can take. In the third position, X = 14.5, Y = 11, but after adding 1.5 brings it to 12.5. Since White is a small favorite, White can Redouble and Black has a clear take.

Be forewarned that this rule is not as accurate as the previous ones, and serious errors are possible. For example, if we tweak the third position so Black has position 4321, then even though White is a small, cubeless favorite, doubling would be a 0.15 error and Black should beaver!

4 checkers versus 3 checkers

Earlier we saw what to do when White has 4 checkers and Black has 3 checkers on the ace point. Specifically, Black could take when White has position 4321 or 3321, but must pass any stronger position. (In such positions, White has APC ≤ 9.) If Black has any other 3 checker position, then Black must pass any White position with APC ≤ 10. But once White has APC > 10, then we apply the previous trick of penalizing White 1.5 pips for the extra checker, and applying the 3 vs 3 rules. In other words, if White has APC = X > 10, then Black's Point of Last Take is X + 2.5, and White's Doubling Point is X – 0.5, and Redoubling point is at X.

3 checkers versus 2 checkers

Now suppose that White has 3 checkers left, and Black has just two checkers left. Should White ever double in those situations? Surely, White should never double if Black has more than 19 winning numbers, since if White does not take all 3 checkers off on the first roll, then Black will redouble White with a strong advantage. But if Black has 19 winning numbers or less, then White can consider doubling.

For example, if White has 3 checkers on the ace point (Position 111) and Black has position 52 with 19 winning numbers, as shown in Figure 8. Here, White is the favorite, since Black gets to roll only 5/6 of the time, and thus has a cubeless winning chance of (5/6)(19/36) = 44%. Even with Black's cube leverage, it can be shown that White still has a Redouble.

But what if White's 3 checkers are not all on the ace point? Or Black is an underdog on roll? The key factors are White's Adjusted Pip Count X and the number of rolls that bear off both of Black's checkers, which we will call B, as shown in Table 2. For example, if Black has position 65, then B = 6, since Black bears off both checkers with rolls 66, 55, 44, 33, and 65. This position almost the same as having three checkers on the ace point, which also has 6 working numbers, but the 65 position is a tad worse, since it is not guaranteed to bear off in two rolls.

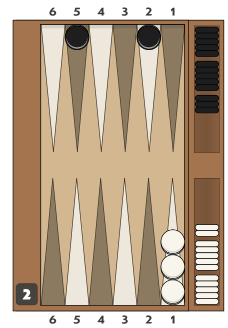

Figure 8. With White on Roll, the cube action is Redouble/Take.

Table 2. Two checker positions and the number of rolls B that bear off both checkers.

B	4	6	8	10	11	13	14	15	17	19	23	25	26	29	34	36
Posn	66	65	64	63	44	62	53	61	43	52	51	32	22	41	31	21
		55		54					33		42					11

Naturally, Black will have a Take if White has enough misses (high APC) or Black has enough working numbers (high B), or some combination thereof. As seems to be the case with all 3-checker positions, the number 11 plays a prominent role. Let's call this the Rule of 11.

The Rule of 11: When White has three checkers with APC = X and Black has two checkers which can be born off with B rolls, then Black can Take if and only if:

$$X \geq 11 \text{ or } B \geq 11 \text{ or } X + B \geq 18$$

The only serious exception is the rare situation where Black's has position 66 (where B = 4). Here, Black can take if and only if X ≥ 12. (mnemonic: 6 + 6 = 12).

The first two parts of this rule make sense, since if X ≥ 11, we know from the 11-12 Rule, that Black has a take with three checkers on the ace point, and all of the two checker positions (except for 66) are at least as good as that. Likewise, if B ≥ 11, then Black wins at least (5/6)(11/36) = 55/216 > 25% of the time, so it must be a take.

When should White double? Keeping in mind the restriction that we must have B ≤ 19, our decision rule again depends on the magic number 11.

$$\text{If } X < 11\text{, then White redoubles if and only if } X + B \leq 27.$$

My mnemonic for this is that Black can take when X + B is at least 18 (or B ≥ 11), and White's doubling point is when X + B is at most 27, which is the next multiple of 9.

When X ≥ 11, we know that Black is guaranteed to have a Take. When X = 11 or 12, the doubling points are when B ≤ 13 or B ≤ 9, respectively. This is equivalent to saying X + B ≤ 24 or X + B ≤ 21, respectively. Since the upper bound for X + B drops from 27 to 24 to 21, we can combine these rules (and handle the cases where X = 11.5 or 12.5) by modifying the previous rule as follows.

$$\text{If } 11 \le X \le 12.5,$$
$$\text{White redoubles when}$$
$$X + B + 3(X - 10) \le 27$$

The above equation can also be written as $4X + B \le 57$, but I find the above formula to be easier to remember. Let's apply these rules to the examples in Figure 9.

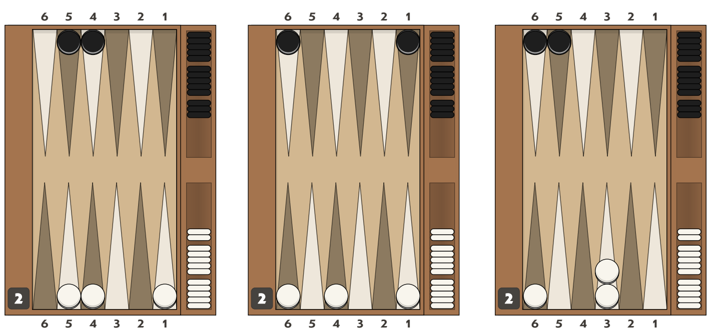

Fig. 9. Determine the cube actions for the above positions

In our first example, White's position 541 has $X = 10$ and Black's position 54 has $B = 10$, so since $X + B = 20$ is between 18 and 27, this position would be a clear Redouble and Take. In the next two examples, White has $X = 11$ and $X = 12.5$, respectively, so the Takes are automatic. Black's position 61 has $B = 15$, and since $X + B + 3 = 29$, White should not double. In the last position, Black's position 65 has $B = 6$, so $X + B + 7.5 = 26$ allows White to Redouble.

When $X = 13$ or 14, White can only double position 66, which may not even be worth remembering. Just don't double with 13 or more pips, just like White would never double with 13 pips in the original 11-12 Rule.

2 checkers versus 3 checkers

As you might imagine, if White has 2 checkers and Black has 3 checkers, it would be very unlikely for Black to take, even if all three are on the ace point. For practical purposes, it is almost always Redouble/Pass and you can leave it at that.

But for those of us who are impractical, the number 11 shows up again. If White has position 65 (11 pips), then Black has a bare take (equity = 0.981) with position 111, but has to drop with anything worse (like 321). If White has position 66, then White can always redouble, and Black can take with APC ≤ 11.

2 checkers versus 2 checkers

We conclude with two checker versus two checker positions. Admittedly, many of these cube decisions can be made over the board, especially the Take/Pass decision, so let's start with the doubling decision with the **40% Rule**.

40% Rule: White should double if White is a favorite to win on the first roll, or Black has less than a 40% chance of winning on their first roll.

For those who prefer formulae, if White has W winning numbers (i.e., numbers that bear off both checkers) and Black has B winning numbers, assuming Black gets to roll, then the 40% Rule says that White should double if $W \ge 19$ or if $(36 - W)B \le 510$ (since $510/1296 = 0.39$).

With rare exceptions (particularly when $W = 17$ or 19 and Black has a very efficient recube), all doubles are redoubles too.

As for the take decision, you could probably just work it out over the board, but the following rule (shown to me by Chuck Bower) is easy.

1. If W ≥ 27, then Black should pass.
2. If W < 27 and B < W, then Black should Pass.
3. If W < 27 and B > W, then Black should Take.
4. If W < 27 and B = W, then Black should Take if B ≥ 17.

To summarize, if it's not an automatic pass (where W ≥ 27), then Black should take if their position is better than White, and should drop if their position is worse. The only exceptions to this rule are 54 versus 44, and 55 versus 64 (which are drops) have error less than 0.015. If the positions are the same, Black takes when B ≥ 17. By this rule, the mirrored positions that are takes are 22, 32, 42, 51, 52, 43, and 33 (although 33 is a very thin pass).

Acknowledgement: I wish to thank Ilia Guzei, David Presser, and Chuck Bower for valuable feedback and my student Mathus Leungpathomaram for invaluable programming assistance. I am grateful to the Sundeman Scholar Fund for supporting this research.

Summary

Let's compactly summarize the results of this paper. We leave out rare exceptions for ease of reference.

Let White be on Roll with APC = X, and Black has APC = Y.

(APC has penalties ½, ½, ½, ½, 1, 2 for each stacked checker on 6, 5, 4, 3, 2, 1, respectively.) Unless explicitly mentioned, all Doubles are Redoubles. Let DTW denote the Double/Take Window.

3 checkers versus 2-roll no-miss position: 11-12 Rule:

DTW: 10.5 ≤ X ≤ 12

(Alternatively, SAPC = 11 or 12)

4 checkers versus 2-roll no-miss position: 4-3-2-1 Rule:

3321 is borderline Take; 5321 is Double but no Redouble.

(2222 or 3331 is a Beaver)

3 checkers versus 3-roll no-miss position on ace:

Redouble/Pass.

3-roll no-miss position versus 3 checkers.

DTW: Y ≥ 15.

4 checkers versus 3-roll no-miss: The Liesl Rule:

DTW: 16 ≤ X ≤ 18.

(almost as accurate: Pip Count = 16 or 17)

3-roll no-miss position versus 4 checkers: The Louisa Rule:

DTW: 12.5 ≤ Y ≤ 15.

2-Roll Fail Probability. If White has 3 checkers, then for X ≥ 9,

P(2-roll Fail) ≈ 8X – 70.

Any 3 checkers vs any 3 checkers (not no-miss): 3-v-3 Rules:

If X < 11, Redouble/Pass
If X = 11 or 11.5, DTW: Y < X.
If X = 12 or 12.5, DTW: Y ≤ X (rounded up).
If 13 ≤ X ≤ 14.5, DTW: X – 3 ≤ Y ≤ X + 1.
If X ≥ 15, DTW: X – 2 ≤ Y ≤ X + 1.

3 checkers vs 4 checkers or 4 checkers vs 3 checkers:

Apply 1.5 pip penalty to the 4 checkers then apply 3-v-3 Rules.

2 checkers versus 3 checkers:

Redouble/Pass

For 2-checker positions, let Black have B rolls that bear off both checkers and White have W rolls that bear off both checkers.

3 checkers versus 2 checkers (with $B \le 19$): 11 Rule

Black takes when $X \ge 11$ or $B \ge 11$ or $X + B \ge 18$.
For $X < 11$, DTW: $18 \le X + B \le 27$
For $11 \le X \le 12.5$, White doubles when $X + B + 3(X - 10) \le 27$
For $X \ge 13$, No Double/Take.

2 checkers versus 2 checkers: 40% Rule:

White doubles if $W \ge 19$ or $(36 - W)B \le 510$.
If $W \ge 27$, then Black Passes.
If $W < 27$, then Black Takes if $B > W$ or if $17 \le B = W < 27$.

References

Danny Kleinman, "Benjamin's Benchmarks", *But Only the Hogs Win Backgammons*, self-published, Los Angeles, 1991, p. 70.